**EXISTENCE OF MULTIPLE WEAK SOLUTIONS FOR A CLASS**

**OF DISCRETE BOUNDARY VALUE PROBLEMS**

**VIA VARIATIONAL METHODS**

SỰ TỒN TẠI ĐA NGHIỆM YẾU CHO MỘT LỚP BÀI TOÁN GIÁ TRỊ

BIÊN RỜI RẠC BẰNG PHƯƠNG PHÁP BIẾN PHÂN

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***ABSTRACT****: In this paper, we consider a class of discrete boundary value problems. Under some suitable conditions on the nonlinearities, we prove the existence of at least three weak solutions for the problem. The approach is essentially based on a variational principle due to Ricceri [15].*

***Keywords:*** *Discrete boundary value problems, three critical points theorem, variational methods*

***TÓM TẮT****:* *Trong bài báo này, chúng tôi xét một lớp bài toán giá trị biên rời rạc. Với một số điều kiện thích hợp được ấn định lên các biểu thức phi tuyến, chúng tôi chứng minh bài toán có ít nhất ba nghiệm yếu. Phương pháp được dùng ở đây chủ yếu dựa trên một nguyên lí biến phân của Ricceri trong bài báo [15].*

***Từ khóa:*** *Bài toán giá trị biên rời rạc, định lí ba điểm tới hạn, phương pháp biến phân*

**1. INTRODUCTION AND PRELIMINARIES**

In the paper, we are interested in the existence of solutions for the following discrete bound-ary value problem(1.1)

where is a positive integer, denotes the discrete interval with and are integers such that , is the forward difference operator, are positive parameters, are two functions with respect to , , and such that

and

It is well known that in various fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others, the mathematical modeling of important questions leads naturally to the consideration of nonlinear difference equations. For this reason, in recent years, many authors have widely developed various methods and techniques, such as fixed points theorems, upper and lower solutions methods, topological methods or variational methods, to study discrete problems, we refer the readers to the interesting book by Agarwal [1] and the papers [6, 14] for the back-ground knowledge on the topic. In [2, 4, 5, 6, 7, 12, 16], the authors use different methods to study the existence and multiplicity of solutions for the discrete boundary value problem

(1.2)

where . In [14], Mihilescu et al. study the eigenvalue problem for anisotropic discrete boundary value problem of the form (1.3)

where . , and is a positive parameter. After the work [14], there have been some papers involving anisotropic discrete boundary value problems, we refer to recent works [3, 8, 9, 11, 13]. Motivated by the papers mentioned above, we study the existence of solutions for system (1.1) with two parameters and . The approach used here is essentially based on a variational principle due to Ricceri [15]. Our result introduced here improves the earlier ones in [7, 8, 13].

Let us define the function space .

Then *W* is a T-dimensional Hilbert space with the inner product

,

and the associated norm is defined by

.

On the other hand, it is useful to introduce other norms on *W*, namely

, .

It can be verified that

, , . (1.4)

**Lemma 1.1** (see [14]).*(i) Let and . Then*

*.*

*(ii) Let and . Then*

*.*

*(iii) For any , there exists a positive constant such that*

*,.*

Moreover, from (1.4) and Lemma 1.1 (iii), it reads

. (1.5)

We will seek weak solutions of problem (1.1) in the Hilbert space , which is equipped with the inner product

for all and the norm

for all .

We remark that for every , there exists such that . Therefore, since , a straightforward computation gives

,

And by using the discrete Hlder inequality, we have that

. (1.6)

From this, there exists such that

. (1.7)

The key in our argument is the following result.

**Proposition 1.2** (see [15]). *Let X* *be a separable and reflexive real Banach space; is bounded on each bounded subset of X, continuously Gteaux differentiable and sequentially weakly lower semicontinuous functional whose Gteaux derivative admits a continuous inverse on ; a continuously Gteaux differentianle functional whose Gteaux derivative is compact. Moreover, assume that*

*(i) for all and that there are , such that*

*(ii)*

*(iii).*

*Then, there exist an open interval and a positive real number , such that for every , and every continuously Gteaux differentiable functional with compact derivative, there exists such that for each , the equation*

*has at least three solutions whose norms are less than .*

**2. MAIN RESULT**

In this section, we shall state and prove the main result of this paper. We use the letter to denote general positive constant whose value may change from line to line. From now on, we assume the following hypotheses hold:

(F0) for all ;

(F1) There exists and functions such that

and for all ,

, ,

where and ;

(F2) There exist and such that for all and for all .

(G0) is assumed to be a measurable function with respect to and is a function with respect to for all and satisfies

for all and and .

It should be noticed that there are many functions and satisfying the hypotheses (F0)-(F2) and (G0). Indeed, let

and

,

where , , , and .

Then,

,

,

and we can verify the hypotheses (F0)-(F2) and (G0). We make the definition of weak solutions for problem (1.1) as follows.

**Definition 2.1.** We say that is a weak solution of problem (1.1) if

for all .

The main result of this paper can be formulated in the following theorem.

**Theorem 2.2.** *Assume that is a Gteaux differentiable function with respect to and satisfies the conditions (F0)-(F2). Then there exist an open interval of and a positive real number such that, for every and every Gteaux differentiable function with respect to satisfying (G0), there exists such that for each problem (1.1) has at least three weak solutions whose norms in X are less than .*

Our idea for proving Theorem 2.2 is to apply Proposition 1.2. For this reason, let us define the functionals as follows:

,

,

. (2.1)

It is well-known that and *J* are well-defined and continuously Gâteaux differentiable with

and

for all .

**Lemma 2.3.** *The functional is sequentially weakly lower semicontiguous, bounded on each bounded subset of . Moreover, admits a continuous inverse on the dual space of .*

***Proof.*** Since is of class C1 and coercive on Hilbert space , and is finite-dimensional, it follows that is weakly lower semicontiguous (see [10]).

Let us proceed for the boundedness of . Let be a bounded subset of . For any , we can consider and . Thus, from (1.5) there holds

)

.

which shows the boundedness of on . We continue to show the existence of the inverse function . To this end, let us show the strict monotonicity of . For the case , we have

]

.

By the well-known inequality, for any , we have for all

Then, we get

Therefore us strictly monotone, which means is an injection. Moreover, let . Then

as ,

that is, is coercive. Then by Minty-Browder theorem (see [17]), we obtain that is a surjection. As a consequence, has an inverse mapping . It only remains to show that is continuous. To this end, let withand let . Then, and which means that is bounded in . The space , being a Hilbert space, is reflexive. Therefore, by Banach-Alaoglu theorem the sequence is relatively compact in endowed with the weak topology. This means that there exist and a subsequence, again denoted by , such that in . Considering the fact that is a finite-demensioanal Hibert space, it reads in , that is, in since the limit is unique. Therefore is continuous.

*Proof of Theorem 2.2*. By Lemma 2.3, is sequentially weakly lower semiconttinuous, bounded on each bounded subset of admits a continuous inverse on the dual space . Moreover, by (F1), *J’* is compact. In order to apply Proposition 1.2, we need to verify that the assumptions *(i)-(iii)* hold.

Now, we first prove that

for all . Indeed, and without loss of generality, we will distinguish two case since is considered to tend to infinity. We will restrict ourselves to the cases when , is bounded and when and prove the coercivity of . In view of condition (F1), there exist and such that

We also deduce from (1.6) that

, ,

(2.2)

In view of condition (F1), we obtain that

(2.3)

for all and all . If is bounded then Young’s inequality and relations (2.2), (2.3) yields

We also have

From the above information, it follows the coercivity of since and is bounded.

In the case , we have

Remark that

Then the coercivity of can be easily deduced since and and Hence, *(i)* of Proposition 1.2 is verified.

Now, in view of condition (F0) we have *F(k,0,0) = 0* for all and there exists such that

Let *a, b* be two real numbers such that with *L* given by (1.7), and

Then we obtain

Define the real number

(2.4)

We consider uch that and for every we have

and

Then clearly we have

and *(ii)* is verified.

On the other hand, we have

Now, we investigate the case for .By Lemma 1.1 we have

From this, we have

Taking into account that

(2.5) (2.6)

for all and all such that with *L* given by (1.7)*.* It follows from relations (2.5) and (2.6) that

Consequently, we obtain

.

Then the condition *(iii)* of Proposition 1.2 is satisfied.

Moreover, since the function is measurable and in satisfying the condi-tion (G0), then the functional is well defined and continuously Gteaux differentiable on *X*, with compact derivative, and one has

for all . So, in view of ***Proposition 1.2***, problem (1.1) has at least three weak solutions which are critical points of functional . The proof of Theorem 2.2. is completed.

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